Journal of Statistical Physics, Vol. 124, No. 6, September 2006 (© 2006)
DOI: 10.1007/s10955-006-9159-y

Overcrowding estimates for zeroes of Planar and Hyperbolic Gaussian analytic functions[1]

Manjunath Krishnapur[2]

Received November 17, 2005; accepted June 21, 2006
Published Online: August 5, 2006

We consider the point process of zeroes of certain Gaussian analytic functions and find the asymptotics for the probability that there are more than m points of the process in a fixed disk of radius r, as $m \to \infty$. For the planar Gaussian analytic function, $\sum_{n\geq 0} \frac{a_n z^n}{\sqrt{n!}}$, we show that this probability is asymptotic to $e^{-\frac{1}{2}m^2 \log(m)}$. For the hyperbolic Gaussian analytic functions, $\sum_{n\geq 0} \binom{-\rho}{n}^{1/2} a_n z^n$, $\rho > 0$, we show that this probability decays like e^{-cm^2}.

In the planar case, we also consider the problem posed by Mikhail Sodin[(1)] on moderate and very large deviations in a disk of radius r as $r \to \infty$. We partially solve the problem by showing that there is a qualitative change in the asymptotics of the probability as we move from the large deviation regime to the moderate.

KEY WORDS: random analytic functions, chaotic analytic zero points, CAZP, large deviations, fluctuations, coulomb gas, one component plasma

1. INTRODUCTION

In this paper we consider the following Gaussian analytic functions (GAFs):

- **Planar GAF :** Often called the *Chaotic analytic function* in the Physics literature, this is the random analytic function

$$\mathbf{g}(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{\sqrt{n!}}$$

where a_n are i.i.d. standard complex Gaussian random variables (i.e., a_n has pdf $\frac{1}{\pi}e^{-|z|^2}$). This defines an entire function (almost surely).

[1] Research supported by NSF grant #DMS-0104073 and NSF-FRG grant #DMS-0244479.
[2] Department of Statistics, U.C. Berkeley, CA 94720, USA; e-mail: manju@stat.berkeley.edu

- **Hyperbolic GAFs :** For each $\rho > 0$ let

$$\mathbf{f}_\rho(z) = \sum_{n=0}^{\infty} \binom{-\rho}{n}^{1/2} a_n z^n$$

where as before a_n are i.i.d. standard complex Gaussians. Almost surely, $\mathbf{f}_\rho$ is an analytic function in the unit disk (and no more).

These particular GAFs are of interest because the distributions of their zero sets are invariant under isometries of the Euclidean plane and isometries of the Hyperbolic plane respectively. In particular, the zero set of $\mathbf{f}_\rho$ has constant intensity $\frac{\rho}{\pi}$ (w.r.t. $\frac{|dz|^2}{(1-|z|^2)^2}$) and is the only zero set of a GAF that is conformally invariant in the unit disk and has this density. See Sodin and Tsirelson[(2)] and Sodin[(3)] for proofs of these assertions. In the planar case too one can define GAFs with invariant zero distribution of intensity ρ for any $\rho > 0$, but these are just scaled versions of the zero set of $\mathbf{g}$ defined above. The zero set of $\mathbf{g}$ has intensity $\frac{1}{\pi}$ w.r.t. Lebesgue measure on the plane (and again is the only GAF zero set with this intensity).

We denote the zero set by $\mathcal{Z}$. Let $n(r)$ denote the number of points of $\mathcal{Z}$ in the disk of radius r around 0 (The GAF will be clear from the context). We address the following two problems.

1. **Overcrowding:** Yuval Peres asked the following question and conjectured that the probability decays as $e^{-cm^2 \log(m)}$ in the planar case (personal communication).
   Question: Fix $r > 0$, ($r < 1$ in the Hyperbolic case). Estimate $\mathbf{P}[n(r) > m]$ as $m \to \infty$.

   One motivation for such a question is in Fig. 1. There one can see the distribution of the zero process under certain conditions on the coefficients that force a large number of zeroes in the disk of radius 2 (this is *not* the zero set conditioned to have overcrowding—that seems harder to simulate). The picture suggests that the distribution of the conditioned process may be worth studying on its own. A large deviation estimate of the kind we derive will presumably be a necessary step in such investigations.

   The answer is different in the two settings. We prove-

Theorem 1. *Consider the planar GAF* $\mathbf{g}$. *Then for any fixed* $r > 0$, *as* $m \to \infty$, *we have*

$$\log \mathbf{P}[n(r) \geq m] = -\frac{1}{2} m^2 \log(m)(1 + o(1)).$$

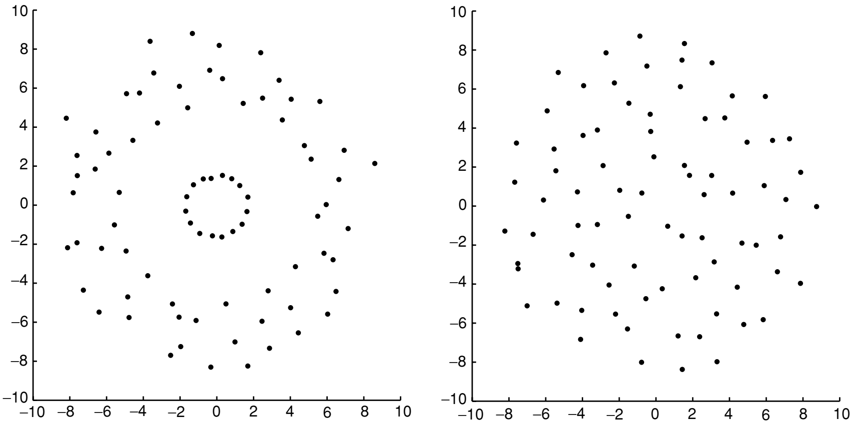

Fig. 1. Left: A sample of zeroes of **g**. Right: A sample of zeroes when the coefficients are conditioned in a specific way to force many zeroes in the disk of radius 2. More precisely, we make the 16th coefficient large, and the first fifteen coefficients small, so that by Rouche's theorem there are at least 16 zeroes in the disk of radius 2.

Theorem 2. *Fix $\rho > 0$ and consider the GAF $\mathbf{f}_\rho$. For any fixed $r < 1$, there are constants β, C_1, C_2(depending on ρ and r) such that for every $m \geq 1$,*

$$C_1(r)e^{-\frac{m^2}{|\log(r)|}} \leq \mathbf{P}[n(r) \geq m] \leq C_2(r)e^{-\beta(r)m^2}.$$

2. **Moderate, large and very large deviations:** Inspired by the results obtained by Jancovici, Lebowitz and Manificat[4] for Coulomb gases in the plane (e.g., Ginibre ensemble), M. Sodin[1] has conjectured the following. **Conjecture:** Let $n(r)$ be the number of zeroes of the planar GAF **g** in the disk $D(0, r)$. Then, as $r \to \infty$

$$\frac{\log\log\left(\frac{1}{\mathbf{P}[|n(r)-r^2|>r^\alpha]}\right)}{\log r} \to \begin{cases} 2\alpha - 1, & \frac{1}{2} \leq \alpha \leq 1; \\ 3\alpha - 2, & 1 \leq \alpha \leq 2; \\ 2\alpha, & 2 \leq \alpha. \end{cases} \tag{1}$$

The idea here is that the deviation probabilities undergo a qualitative change in behaviour when the deviation under consideration becomes comparable to the perimeter ($\alpha = 1$) or to the area ($\alpha = 2$) of the domain.

Sodin and Tsirelson[5] had already settled the case $\alpha = 2$ by showing that for any $\delta > 0$, $\exists c_1(\delta), c_2(\delta)$ such that

$$e^{-c_1(\delta)r^4} \leq \mathbf{P}[|n(r) - r^2| > \delta r^2] \leq e^{-c_2(\delta)r^4}.$$

Here we consider $\mathbf{P}[n(r) - r^2 > r^\alpha]$ and prove that a phase transition in the exponent occurs at $\alpha = 2$. More precisely we prove that the conjecture holds for $\alpha > 2$ and show the lower bound for $1 < \alpha < 2$.

Theorem 3. *Fix $\alpha > 2$. Then*

$$\mathbf{P}\left[n(r) \geq r^2 + \gamma r^\alpha\right] = e^{-\left(\frac{\alpha}{2}-1\right)\gamma^2 r^{2\alpha} \log r(1+o(1))}.$$

Theorem 4. *Fix $1 < \alpha < 2$. Then for any $\gamma > 0$,*

$$\mathbf{P}\left[n(r) \geq r^2 + \gamma r^\alpha\right] \geq e^{-\gamma^3 r^{3\alpha-2}(1+o(1))}.$$

Remark 5. *Nazarov, Sodin and Volberg have recently proved all parts of the conjecture (personal communication).*

We prove Theorem 1 in Sec 2, Theorem 2 in Sec 3, and Theorem 3 and Theorem 4 in Sec 4.

2. OVERCROWDING—THE PLANAR CASE

In this section we prove Theorem 1. Before that we explain why one expects the constant $\frac{1}{2}$ in the exponent in Theorem 1, by analogy with the Ginibre ensemble.

2.1. Ginibre Ensemble

The Ginibre ensemble is the determinantal point process (see Ref. 6 or 7 for definitions) in the plane with kernel

$$K(z, w) = \frac{1}{\pi} e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|w|^2 + z\bar{w}}. \tag{2}$$

This process is of interest because it is the limit in distribution, as $n \to \infty$, of the point process of eigenvalues of an $n \times n$ matrix with i.i.d. standard complex Gaussian entries.[8]

The Ginibre ensemble has many similarities to the zero set of $\mathbf{g}$. In particular, the Ginibre ensemble is invariant in distribution under Euclidean motions, has constant intensity $\frac{1}{\pi}$ in the plane and has the same negative correlations as $\mathcal{Z}_\mathbf{g}$ at short distances. Therefore there are other similarities too, for instance, see Ref. 9. There are also differences between the two point processes. For instance, the Ginibre ensemble has all correlations negative, whereas for the zero set of $\mathbf{g}$, long-range two-point correlations are positive. However, in our problem, since we are considering a fixed disk and looking at the event of having an excess of zeroes in it, it seems reasonable to expect the same behaviour for both these point processes, since it is the short range interaction that is relevant. In case of the Ginibre ensemble, the overcrowding problem is easy to solve.

Theorem 6. *Let $n_G(r)$ be the number of points of the Ginibre ensemble in the disk of radius r around 0 (by translation invariance, the same is true for any disk of radius r). Then for a fixed $r > 0$,*

$$\mathbf{P}\left[n_G(r) \geq m\right] = e^{-\frac{1}{2}m^2 \log(m)(1+o(1))}.$$

Proof: By Kostlan,[10] the set of absolute values of the points of the Ginibre ensemble has the same distribution as the set $\{R_1, R_2, \ldots\}$, where R_n are independent, and R_n^2 has Gamma$(n, 1)$ distribution for every n. Hence $R_n^2 \stackrel{d}{=} \xi_1 + \cdots + \xi_n$, where ξ_k are i.i.d. Exponential random variables with mean 1, and it follows that

$$\mathbf{P}\left[R_n^2 < r^2\right] \geq \prod_{k=1}^{n} \mathbf{P}\left[\xi_k < \frac{r^2}{n}\right] \geq \left(\frac{r^2}{2n}\right)^n,$$

as long as $n \geq r^2$, because $\mathbf{P}\left[\xi_1 < x\right] \geq \frac{x}{2}$ for $x < 1$. Therefore we get

$$\mathbf{P}\left[n_G(r) \geq m\right] \geq \prod_{n=1}^{m} \mathbf{P}\left[R_n^2 < r^2\right] \tag{3}$$

$$\geq \prod_{n=1}^{m} \left(\frac{r^2}{2n}\right)^n \tag{4}$$

$$= \left(\frac{r^2}{2}\right)^{\frac{m(m+1)}{2}} e^{-\sum\limits_{n=1}^{m} n \log(n)}. \tag{5}$$

Here and elsewhere we shall encounter the term $\sum_{n=1}^{m} n \log(n)$. We compute its asymptotics now.

$$n \log(n) \leq x \log(x) \leq (n+1) \log(n+1) \qquad \text{for } n \leq x \leq n+1$$

Integrate from 1 to $m+1$ and note that

$$\int_1^a x \log(x)\, dx = \frac{1}{2} a^2 \log(a) - \frac{a^2}{4} + \frac{1}{4},$$

to get

$$\sum_{n=1}^{m} n \log(n) \leq \frac{1}{2}(m+1)^2 \log(m+1) - \frac{(m+1)^2}{4} + \frac{1}{4} \leq \sum_{n=1}^{m+1} n \log(n). \tag{6}$$

Thus (5) gives

$$\mathbf{P}\left[n_G(r) \geq m\right] \geq e^{-\frac{1}{2}(m+1)^2 \log(m+1)+\frac{(m+1)^2}{4}-\frac{1}{4}+\frac{m(m+1)}{2}\log(r^2/2)}$$
$$= e^{-\frac{1}{2}m^2 \log(m)+O(m^2)}.$$

To prove the inequality in the other direction, note that

$$\mathbf{P}\left[n_G(r) \geq m\right] \leq \mathbf{P}\left[\sum_{n=1}^{m^2} \mathbf{1}\left(R_n^2 < r^2\right) \geq m\right] + \sum_{n=m^2+1}^{\infty} \mathbf{P}\left[R_n^2 < r^2\right]$$
$$\leq \binom{m^2}{m} \prod_{n=1}^{m} \mathbf{P}\left[R_n^2 < r^2\right] + \sum_{n>m^2} e^{-n\log(n)(1+o(1))}.$$

In the second line, for the first summand we used the fact that R_n^2 are stochastically increasing and for the second term we used the well-known fact $\mathbf{P}[R_n^2 < r^2] = \mathbf{P}\left[\mathrm{Pois}(r^2) \geq n\right]$ and then the usual bound on the tail of a Poisson random variable, namely $\mathbf{P}\left[\mathrm{Poisson}\,(\theta) \geq a\right] \leq e^{-a\log(a/\theta)+a-\theta}$.

Using the same idea to bound $\mathbf{P}\left[R_n^2 < r^2\right]$ in the first summand, we obtain

$$\mathbf{P}\left[n_G(r) \geq m\right] \leq \binom{m^2}{m} \prod_{n=1}^{m} e^{-n\log(n/r^2)-r^2+n} + e^{-m^2\log(m^2)(1+o(1))}$$
$$\leq \binom{m^2}{m} e^{\frac{m(m+1)}{2}(1+\log(r^2))-mr^2-\sum_{n=1}^{m} n\log(n)} + e^{-m^2\log(m^2)(1+o(1))}$$
$$= e^{-\frac{1}{2}m^2\log(m)(1+o(1))} \qquad \text{(using (6) again)}.$$

In the last line we used $\binom{m^2}{m} < m^{2m}$. This completes the proof. □

2.2. Proof of Theorem 1

Our method of proof is largely based on that of Sodin and Tsirelson.[5] (They estimate the "hole probability," $\mathbf{P}\left[n(r) = 0\right]$ as $r \to \infty$.)

Proof of Theorem 1: Lower bound. Suppose the mth term dominates the sum of all the other terms on $\partial D(0;r)$, i.e., suppose

$$\left|\frac{a_m z^m}{\sqrt{m!}}\right| \geq \left|\sum_{n\neq m} \frac{a_n z^n}{\sqrt{n!}}\right| \qquad \text{whenever } |z| = r. \tag{7}$$

Then, by Rouche's theorem $\mathbf{g}(z)$ and $\frac{a_m z^m}{\sqrt{m!}}$ have the same number of zeroes in $D(0;r)$. Hence $n(r) = m$. Now we want to find a lower bound for the probability of the event in (7). Note that the left side of (7) is identically equal to $\frac{|a_m|r^m}{\sqrt{m!}}$.

Now suppose the following happen-

1. $|a_n| \le n \ \forall n \ge m+1$.
2. $|a_m| \ge (\alpha+1)m$ where α will be chosen shortly.
3. $|a_n|\frac{r^n}{\sqrt{n!}} < \frac{r^m}{\sqrt{m!}}$ for every $0 \le n \le m-1$.

Then the right hand side of (7) is bounded by

$$\begin{aligned} \text{RHS of (7)} &\le \sum_{n=0}^{m-1} |a_n| \frac{r^n}{\sqrt{n!}} + \sum_{n=m+1}^{\infty} \frac{|a_n| r^n}{\sqrt{n!}} \\ &\le \sum_{n=0}^{m-1} \frac{r^m}{\sqrt{m!}} + \sum_{n=m+1}^{\infty} \frac{n r^n}{\sqrt{n!}}. \end{aligned}$$

Remember that r is fixed and hence the summands in the second sum decrease (faster than) geometrically. Therefore, up to a constant, the first term dominates the sum. We obtain,

$$\begin{aligned} \text{RHS of (7)} &\le m\frac{r^m}{\sqrt{m!}} + C\frac{mr^m}{\sqrt{m!}} \\ &= (C+1)m\frac{r^m}{\sqrt{m!}} \\ &\le |a_m|\frac{r^m}{\sqrt{m!}} \end{aligned}$$

if $\alpha = C$. Thus if the above three events occur with $\alpha = C$, then the mth term dominates the sum of all the other terms on $\partial D(0;r)$. Also these events have probabilities as follows.

1. $\mathbf{P}[|a_n| \le n \ \forall n \ge m+1] \ge 1 - \sum_{n=m+1}^{\infty} e^{-n^2} \ge 1 - C'e^{-m^2}$.
2. $\mathbf{P}[|a_m| \ge (C+1)m] = e^{-(C+1)^2 m^2}$.
3. The third event has probability as follows. Recall again that $\mathbf{P}[\xi < x] \ge \frac{x}{2}$ if $x < 1$ and ξ is Exponential with mean 1. We apply this below with $x = (\frac{r^{m-n}\sqrt{n!}}{\sqrt{m!}})^2$. This is clearly less than 1 if $n \ge r^2$. Therefore if m is sufficiently large it is easy to see that for all $0 \le n \le m-1$, the same is valid. Thus

$$\mathbf{P}\left[|a_n| \le \frac{r^{m-n}\sqrt{n!}}{\sqrt{(m!)}} \quad \forall n \le m-1\right] = \prod_{n=0}^{m-1} \mathbf{P}\left[|a_n| \le \frac{r^{m-n}\sqrt{n!}}{\sqrt{(m!)}}\right]$$

$$\geq \prod_{n=0}^{m-1} \frac{r^{2m-2n} n!}{2(m!)}$$
$$= r^{m(m+1)} e^{\frac{1}{2}m^2 \log(m)+O(m^2)} 2^{-m} e^{-m^2 \log(m)+O(m^2)}$$
$$= e^{-\frac{1}{2}m^2 \log(m)+O(m^2)}.$$

Since these three events are independent, we get the lower bound in the theorem.

Upper bound. By Jensen's formula, for any $R > r$ we have

$$n(r) \log\left(\frac{R}{r}\right) \leq \int_r^R \frac{n(u)}{u} du = \int_0^{2\pi} \log \left|\mathbf{g}(Re^{i\theta})\right| \frac{d\theta}{2\pi} - \int_0^{2\pi} \log \left|\mathbf{g}(re^{i\theta})\right| \frac{d\theta}{2\pi}. \quad (8)$$

Let $R = R_m = \sqrt{m}$. Sodin and Tsirelson[5] show that

$$\mathbf{P}\left[\log M(t) \geq \left(\frac{1}{2} + \epsilon\right) t^2\right] \leq e^{-e^{\epsilon t^2}} \quad (9)$$

where $M(t) = \max\{|\mathbf{g}(z)| : |z| \leq t\}$.

Now suppose $n(r) \geq m$ and $\log M(R_m) \leq \left(\frac{1}{2} + \epsilon\right) m$ for some $\epsilon > 0$. Then by (8) we have

$$-\int_0^{2\pi} \log |\mathbf{g}(re^{i\theta})| \frac{d\theta}{2\pi} \geq m \log\left(\frac{\sqrt{m}}{r}\right) - \left(\frac{1}{2} + \epsilon\right) m$$
$$= \frac{1}{2} m \log(m) - m \log(r) - \left(\frac{1}{2} + \epsilon\right) m$$
$$= \frac{1}{2} m \log(m) - O(m)$$

Thus

$$\mathbf{P}[n(r) \geq m] \leq \mathbf{P}\left[\log M(R_m) \geq \left(\frac{1}{2} + \epsilon\right) m\right]$$
$$+ \mathbf{P}\left[-\int_0^{2\pi} \log |\mathbf{g}(re^{i\theta})| \frac{d\theta}{2\pi} \geq \frac{1}{2} m \log(m) - O(m)\right]$$
$$\leq e^{-e^{\epsilon m}} + \mathbf{P}\left[-\int_0^{2\pi} \log |\mathbf{g}(re^{i\theta})| \frac{d\theta}{2\pi} \geq \frac{1}{2} m \log(m)(1 + o(1))\right] \text{by } (9).$$

From Lemma 7, we deduce that for any $\delta > 0$, there is a constant C_2 such that

$$\mathbf{P}\left[-\int_0^{2\pi} \log|\mathbf{g}(re^{i\theta})|\frac{d\theta}{2\pi} \geq \frac{1}{2}m\log(m)(1+o(1))\right] \leq C_2 e^{-(2-\delta)(\frac{m}{2}\log(m))^2/\log(\frac{m}{2}\log(m))}$$

$$\leq C_2 e^{-(\frac{1}{2}-\frac{\delta}{4})m^2\log(m)(1+o(1))}.$$

From this, the upper bound follows. □

Lemma 7. *For any given $\delta > 0$, $\exists C_2$ such that*

$$\mathbf{P}\left[-\int_0^{2\pi} \log|\mathbf{g}(re^{i\theta})|\frac{d\theta}{2\pi} \geq m\right] \leq C_2 e^{-\frac{(2-\delta)m^2}{\log(m)}} \forall m.$$

Proof: Let P be the Poisson kernel on $D(0;r)$. Fix $\epsilon > 0$ and let $A_\epsilon = \sup\{P(re^{i\theta}, w) : |w| = \epsilon, \theta \in [0, 2\pi)\}$ and $B_\epsilon = \inf\{P(re^{i\theta}, w) : |w| = \epsilon, \theta \in [0, 2\pi)\}$. Since $\log|\mathbf{g}|$ is a sub-harmonic function, for any w with $|w| = \epsilon$, we get

$$\begin{aligned}
\log|\mathbf{g}(w)| &\leq \int_0^{2\pi} \log|\mathbf{g}(re^{i\theta})| P(re^{i\theta}, w)\frac{d\theta}{2\pi} \\
&\leq A_\epsilon \int_0^{2\pi} \log_+ \left|\mathbf{g}(re^{i\theta})\right|\frac{d\theta}{2\pi} - B_\epsilon \int_0^{2\pi} \log_- \left|\mathbf{g}(re^{i\theta})\right|\frac{d\theta}{2\pi} \\
&= A_\epsilon \int_0^{2\pi} \log_+ \left|\mathbf{g}(re^{i\theta})\right|\frac{d\theta}{2\pi} \\
&\quad + B_\epsilon \left(\int_0^{2\pi} \log\left|\mathbf{g}(re^{i\theta})\right|\frac{d\theta}{2\pi} - \int_0^{2\pi} \log_+ \left|\mathbf{g}(re^{i\theta})\right|\frac{d\theta}{2\pi}\right) \\
&\leq B_\epsilon \int_0^{2\pi} \log\left|\mathbf{g}(re^{i\theta})\right|\frac{d\theta}{2\pi} + A_\epsilon \log_+ M(r).
\end{aligned}$$

This implies $\log M(\epsilon) \leq B_\epsilon \int_0^{2\pi} \log|\mathbf{g}(re^{i\theta})|\frac{d\theta}{2\pi} + A_\epsilon \log_+ M(r)$.

Therefore if $\int_0^{2\pi} \log|\mathbf{g}(re^{i\theta})|\frac{d\theta}{2\pi} \leq -m$, then one of the following must happen. Either $\{\log M(\epsilon) \leq -B_\epsilon m + \sqrt{m}\}$ or $\{A_\epsilon \log_+ M(r) > \sqrt{m}\}$.

Using (9), since $M(r) < M(Cm^{\frac{1}{4}})$ for any C, we see that $\mathbf{P}[\log_+ M(r) > \frac{\sqrt{m}}{A_\epsilon}] \leq e^{-e^{Cm}}$ for some constant C depending on ϵ. Hence

$$\mathbf{P}\left[\int_0^{2\pi} \log|\mathbf{g}(re^{i\theta})|\frac{d\theta}{2\pi} \leq -m\right] \leq e^{-e^{Cm}} + \mathbf{P}\left[\log M(\epsilon) \leq -B_\epsilon m + \sqrt{m}\right]$$

$$\leq e^{-e^{Cm}} + e^{-2B_\epsilon^2 \frac{m^2}{\log(m)}(1+o(1))}$$

where in the last line we have used Lemma 8.

As $\epsilon \to 0$, $B_\epsilon \to 1$ and hence the proof is complete. □

Now we prove the upper bound on the maximum modulus in a disk of radius r that was used in the last part of the proof of Lemma 7. For possible future use we prove a lower bound too.

Lemma 8. *Fix $r > 0$. There are constants α, C_1, C_2 such that*

$$C_1 e^{-\frac{\alpha m^2}{\log(m)}} \leq \mathbf{P}[\log M(r) \leq -m] \leq C_2 e^{-\frac{2m^2}{\log(m)}(1+o(1))}.$$

Proof: **Lower bound.** By Cauchy-Schwarz, $M(r) \leq (\sum_{n=0}^{k-1}|a_n|^2)^{1/2}e^{r^2/2} + \sum_{n=k}^{\infty}\frac{|a_n|r^n}{\sqrt{n!}}$. We shall choose k later. We will bound from below the probability that each of these summands is less than $\frac{e^{-m}}{2}$.

Let ϕ_k denote the density of $\Gamma(k, 1)$.

$$\mathbf{P}\left[\left(\sum_{n=0}^{k-1}|a_n|^2\right)^{1/2} e^{r^2/2} \leq \frac{e^{-m}}{2}\right] = \mathbf{P}\left[\sum_{n=0}^{k-1}|a_n|^2 \leq \frac{e^{-2m}e^{r^2}}{4}\right]$$

$$\geq \phi_k\left(\frac{e^{-2m}e^{r^2}}{8}\right)\frac{e^{-2m}e^{r^2}}{8}$$

$$= e^{-2mk - k\log(k) + O(k)}$$

Also if $|a_n| \leq n^2\ \forall n \geq k$, then the second summand

$$\sum_{n=k}^{\infty}|a_n|\frac{r^n}{\sqrt{n!}} \leq C\frac{r^k k^2}{\sqrt{k!}} \leq Ce^{-k\log(k)/3}$$

Also the event $\{|a_n| \leq n^2\ \forall n \geq k\}$ has probability at least $1 - \sum_{n=k+1}^{\infty} e^{-n^4} \geq 1 - Ce^{-k^4}$.

Thus if we set $k = \frac{\gamma m}{\log(m)}$ for a sufficiently large γ, then both the terms are less than $e^{-\frac{m}{2}}$ with probability at least $e^{-2\gamma m^2/\log(m)}$.

Upper bound. By Cauchy's theorem,

$$a_n = \frac{\sqrt{n!}}{2\pi i} \int\limits_{C_r} \frac{\mathbf{g}(\zeta)}{\zeta^{n+1}} d\zeta,$$

where C_r is the curve $C_r(t) = re^{it}$, $0 \le t \le 2\pi$. Therefore,

$$|a_n| \le \frac{M(r)\sqrt{n!}}{r^n}.$$

Thus we get

$$\mathbf{P}[M(r) \le e^{-m}] \le \prod_{n=0}^{\infty} \mathbf{P}\left[|a_n| \le \frac{e^{-m}\sqrt{n!}}{r^n}\right].$$

$|a_n|^2$ are i.i.d. exponential random variables with mean 1. Therefore, $\mathbf{P}[|a_n| \le \frac{e^{-m}\sqrt{n!}}{r^n}] \le \frac{e^{-2m}n!}{r^{2n}}$. Using this bound for $n \le k := \frac{\beta m}{\log(m)}$, we get

$$\begin{aligned}
\mathbf{P}[M(r) \le e^{-m}] &\le \prod_{n=0}^{k} \frac{e^{-2m}n!}{r^{2n}} \\
&\le Ce^{-2mk + \frac{k^2}{2}\log(k) + O(k^2)} \\
&\le Ce^{\left(-2\beta + \frac{\beta^2}{2}\right)\frac{m^2}{\log(m)} + O\left(\frac{m^2}{(\log(m))^2}\right)}.
\end{aligned}$$

$-2\beta + \frac{\beta^2}{2}$ is minimized when $\beta = 2$ and we get,

$$\mathbf{P}[M(r) \le e^{-m}] \le e^{-2\frac{m^2}{\log(m)}(1+o(1))}. \tag{10}$$

3. OVERCROWDING—THE HYPERBOLIC CASE

3.1. Case $\rho = 1$

We give a quick proof of Theorem 2 in the special case $\rho = 1$, as it is much easier and moreover we get matching upper and lower bounds. The proof is similar to the case of the Ginibre ensemble dealt with in Theorem 6 and is based on the fact that the set of absolute values of the zeroes of $\mathbf{f}_1$ is distributed the same as a certain set of independent random variables. The reason for this similarity between the two cases owes to the fact that both of them are determinantal. The zero set of

$\mathbf{f}_1$ is a determinantal process with the Bergman kernel for the unit disk, namely

$$K_B(z, w) = \frac{1}{\pi} \frac{1}{(1 - z\bar{w})^2},$$

as discovered by Peres and Virág.[11]

Proof of Theorem 2 for $\rho = 1$

By Peres and Virág,[11] Theorem 2 (ii), the set of absolute values of the zeroes of $\mathbf{f}_1$ has the same distribution as the set $\{U_n^{1/2n}\}$ where U_n are i.i.d. Uniform[0, 1] random variables. Therefore,

$$\begin{aligned} \mathbf{P}[n(r) \geq m] &\geq \prod_{n=1}^{m} \mathbf{P}\left[U_n^{1/2n} < r\right] \\ &= \prod_{n=1}^{m} r^{2n} \\ &= r^{m(m+1)}. \end{aligned}$$

To prove the inequality in the other direction, note that

$$\begin{aligned} \mathbf{P}[n(r) \geq m] &\leq \mathbf{P}\left[\sum_{n=1}^{m^2} \mathbf{1}\left(U_n^{1/2n} < r\right) \geq m\right] + \sum_{n=m^2+1}^{\infty} \mathbf{P}\left[U_n^{1/2n} < r\right] \\ &\leq \binom{m^2}{m} \prod_{n=1}^{m} \mathbf{P}\left[U_n^{1/2n} < r\right] + \sum_{n>m^2} r^{2n} \\ &= \binom{m^2}{m} r^{m(m+1)} + \frac{r^{2m^2+2}}{1-r^2} \\ &= r^{m(m+1)}\left(1 + O\left(e^{m\log(m)}\right)\right). \end{aligned}$$

This completes the proof of the theorem for $\rho = 1$.

3.2. All Values of ρ

Remark: Overall, the idea of proof is the same as that of Theorem 1. However we do not get matching upper and lower bounds in the present case, the reason being that in the hyperbolic analogue of Lemma 8, the leading term in the exponent of the upper bound does depend on r, unlike in the planar case. (An examination of the proof of Theorem 1 reveals that we get a matching upper bound only because replacing r by ϵ does not affect the leading term in the exponent in the upper bound in Lemma 8). However we still expect that the lower bound in Theorem 2 is tight. (See remark after the proof).

Proof of Theorem 2: Lower bound. As before we find a lower bound for the probability that the mth term dominates the rest. Note that if $|z| = r$,

$$\left|\mathbf{f}_\rho(z) - \binom{-\rho}{m}^{1/2} a_m z^m\right| \le \sum_{n=0}^{m-1} |a_n| \binom{-\rho}{n}^{1/2} r^n + \sum_{n=m+1}^{\infty} |a_n| \binom{-\rho}{n}^{1/2} r^n \quad (11)$$

Now suppose the following happen-

1. $|a_n| \le \sqrt{n}\ \forall n \ge m+1$.
2. $|a_m| \ge (\alpha+1)\sqrt{m}$ where α will be chosen shortly.
3. $|a_n|\binom{-\rho}{n}^{1/2} r^n < \frac{1}{\sqrt{m}}\binom{-\rho}{m}^{1/2} r^m$ for every $0 \le n \le m-1$.

Then the right hand side of (3.2) is bounded by

$$\begin{aligned}
\text{RHS of (3.2)} &\le \sum_{n=0}^{m-1} |a_n| \binom{-\rho}{n}^{1/2} r^n + \sum_{n=m+1}^{\infty} |a_n| \binom{-\rho}{n}^{1/2} r^n \\
&\le \sum_{n=0}^{m-1} \frac{1}{\sqrt{m}} \binom{-\rho}{m}^{1/2} r^m + \sum_{n=m+1}^{\infty} \sqrt{n} \binom{-\rho}{n}^{1/2} r^n \\
&\le \sqrt{m}\binom{-\rho}{m}^{1/2} r^m + C\sqrt{m}\binom{-\rho}{m}^{1/2} r^m \qquad \text{for some } C (r \text{ is fixed}) \\
&= (C+1)\sqrt{m}\binom{-\rho}{m}^{1/2} r^m \\
&\le |a_m| \binom{-\rho}{m}^{1/2} r^m
\end{aligned}$$

if $\alpha = C$. Thus if the above three events occur with $\alpha = C$, then the mth term dominates the sum of all the other terms on $\partial D(0;r)$. Also these events have probabilities as follows.

1. $\mathbf{P}[|a_n| \le \sqrt{n}\ \forall n \ge m+1] \ge 1 - \sum_{n=m+1}^{\infty} e^{-n} \ge 1 - C' e^{-m}$.
2. $\mathbf{P}[|a_m| \ge (\alpha+1)\sqrt{m}] = e^{-(\alpha+1)^2 m}$.
3. The third event has probability as follows. Recall again that $\mathbf{P}\,[\xi < x] \ge \frac{x}{2}$ if $x < 1$ and ξ is Exponential with mean 1. We apply this below with

$x = \left(\frac{\binom{-\rho}{m}^{1/2} r^{m-n}}{\sqrt{m}\binom{-\rho}{n}^{1/2}} \right)^2$. This is clearly less than 1. Thus

$$\begin{aligned}
&\mathbf{P}\left[|a_n| \le \frac{\binom{-\rho}{m}^{1/2} r^{m-n}}{\sqrt{m}\binom{-\rho}{n}^{1/2}} \quad \forall n \le m-1 \right] \\
&= \prod_{n=0}^{m-1} \mathbf{P}\left[|a_n| \le \frac{\binom{-\rho}{m}^{1/2} r^{m-n}}{\sqrt{m}\binom{-\rho}{n}^{1/2}} \right] \\
&\ge \prod_{n=0}^{m-1} \frac{\binom{-\rho}{m} r^{2m-2n}}{2m\binom{-\rho}{n}} \\
&= r^{m(m+1)} m^{-m} \prod_{n=0}^{m-1} \frac{(m+1)\dots(m+\rho-1)}{(n+1)\dots(n+\rho-1)} \\
&\ge r^{m(m+1)} m^{-m} \prod_{n=0}^{m-1} \frac{m^\rho}{(n+\rho)^\rho} \\
&\ge r^{m(m+1)+O(m\log(m))}.
\end{aligned}$$

Since these three events are independent, we get the lower bound in the theorem.

Upper bound. The proof will proceed along the same lines as in Theorem 1. We need the following analogue of Lemma 8.

Lemma 9. *Fix $r < 1$. Let $M(r) = \sup_{z \in D(0;r)} |\mathbf{f}_\rho(z)|$. Then*

$$\mathbf{P}[M(r) \le e^{-m}] \le e^{-\frac{m^2}{|\log(r)|}(1+o(1))}.$$

Proof: By Cauchy's theorem, for every $n \ge 0$,

$$a_n \binom{-\rho}{n}^{1/2} = \frac{1}{2\pi i} \int_{rT} \frac{\mathbf{f}(\zeta)}{\zeta^{n+1}} d\zeta.$$

From this we get

$$|a_n|^2 \le \frac{M(r)^2}{\binom{-\rho}{n} r^{2n}}.$$

Since $\binom{-\rho}{n} \geq \frac{n^{\rho-1}}{\Gamma(\rho+1)}$, we obtain

$$\begin{aligned}
\mathbf{P}[M(r) \leq m] &\leq \prod_n \mathbf{P}\left[|a_n|^2 \leq \frac{\Gamma(\rho+1)e^{-2m}}{n^{\rho-1}r^{2n}}\right] \\
&\leq \prod_{n=0}^{\frac{m}{\log(1/r)}} \frac{\Gamma(\rho+1)e^{-2m}}{r^{2n}n^{\rho-1}} \\
&\leq e^{-\frac{2m^2}{\log(1/r)} - \left(\frac{m}{\log(1/r)}\right)^2 \log(r) + O(m\log(m))} \\
&= e^{-\frac{m^2}{\log(1/r)} + O(m\log(m))}.
\end{aligned}$$

$\square$

Coming back to the proof of the upper bound in the theorem, fix R such that $r < R < 1$. Then by Jensen's formula,

$$n(r)\log\left(\frac{R}{r}\right) \leq \int_r^R \frac{n(u)}{u}du = \int_0^{2\pi} \log|\mathbf{f}(Re^{i\theta})|\frac{d\theta}{2\pi} - \int_0^{2\pi} \log|\mathbf{f}(re^{i\theta})|\frac{d\theta}{2\pi}. \quad (12)$$

Now consider the first summand in the right hand side of (12).

$$\mathbf{P}\left[\int_0^{2\pi} \log|\mathbf{f}(Re^{i\theta})|\frac{d\theta}{2\pi} > \sqrt{m}\right] \leq \mathbf{P}\left[\log M(R) \geq \sqrt{m}\right].$$

Now suppose that $|a_n| < \lambda^n \; \forall n \geq m+1$ where $1 < \lambda < 1/R$. This has probability at least $1 - C_1 e^{-\lambda^{2m}/2}$. Then,

$$\begin{aligned}
M(R) &\leq \sum_{n=0}^{\infty} |a_n| \binom{-\rho}{n}^{1/2} R^n \\
&\leq \left(\sum_{n=0}^{m} |a_n|^2\right)^{1/2} C_R + C_{R'}
\end{aligned}$$

for some constants C_R and $C_{R'}$.

Thus if $M(R) > e^{\sqrt{m}}$ then either $\sum_{n=0}^{m} |a_n|^2 > Ce^{2\sqrt{m}}$ or else $|a_n| > \lambda^n$ for some $n \geq m+1$. Thus

$$\mathbf{P}\left[M(R) > \sqrt{m}\right] \leq e^{-e^{c\sqrt{m}}}.$$

This proves that

$$\mathbf{P}\left[\int_{R\mathbb{T}} \log|\mathbf{f}(Re^{i\theta})|\frac{d\theta}{2\pi} > \sqrt{m}\right] \le e^{-e^{c\sqrt{m}}}.$$

Fix $\delta > 0$ and R close enough to 1 such that $\log(R) > -\delta$. Then with probability $\ge 1 - e^{-e^{c\sqrt{m}}}$, we obtain from (12),

$$-\int_{r\mathbb{T}} \log|\mathbf{f}(re^{i\theta})|\frac{d\theta}{2\pi} \ge m\left(\log\left(\frac{1}{r}\right) - \delta\right) - \sqrt{m}.$$

Now the calculations in the proof of Lemma 7 show that

$$\log M(\epsilon) \le B_\epsilon \int_0^{2\pi} \log|\mathbf{f}(re^{i\theta})|P(re^{i\theta}, w)\frac{d\theta}{2\pi} + A_\epsilon \log_+ M(r).$$

Here $0 < \epsilon < r$ is arbitrary and A_ϵ, B_ϵ are as defined in Lemma 7. By the same computations as in that Lemma, we obtain, we obtain the inequality

$$\mathbf{P}\left[\int_0^{2\pi} \log|\mathbf{f}(re^{i\theta})|\frac{d\theta}{2\pi} \le -m(|\log r| - \delta) + \sqrt{m}\right] \le e^{-B_\epsilon^2 \frac{m^2\log^2(r)(1-\delta)}{|\log(\epsilon)|}} + e^{-e^{cm}}.$$

Therefore, by (12)

$$\mathbf{P}\left[n(r) \ge m\right] \le e^{-\kappa m^2 \log^2(r)(1+o(1))},$$

where $\kappa = \sup\left\{\frac{B_\epsilon^2}{|\log(\epsilon)|} : 0 < \epsilon < r\right\}$. However it is clear that this cannot be made to match the lower bound by any choice of ϵ. $\square$

Remark : If we could prove

$$\mathbf{P}\left[\int_0^{2\pi} \log|\mathbf{f}(re^{i\theta})|\frac{d\theta}{2\pi} \le -x\right] \le e^{-\frac{x^2}{|\log(r)|}},$$

that would have given us a matching upper bound. Now, one way for the event $\int_0^{2\pi} \log|\mathbf{f}(re^{i\theta})|\frac{d\theta}{2\pi} \le -x$ to occur is to have $\log M(r) < -x$ which, by Lemma 9 has probability at most $e^{-x^2/\log(\frac{1}{r})}$. One way to proceed could be to show that if the integral is smaller than $-x$, so is $\log M(s)$ for s arbitrarily close to r (with high probability). Alternately, if we could bound the coefficients directly by the bound on the integral (as in Lemma 9), that would also give us the desired bound. For these reasons, and keeping in mind the case $\rho = 1$, where we do have a matching upper bound, we believe that the lower bound in Theorem 2 is tight.

4. MODERATE AND VERY LARGE DEVIATIONS FOR THE PLANAR GAF

In this section we prove Theorem 3 and Theorem 4.

Remark 10. *In the case $\alpha \geq 2$, one side of the estimate as asked for in the conjecture (with* $\log\log$ *of the probability) follows trivially from the results in Sodin and Tsirelson.*[(5)] *They prove that for any $\delta > 0$, there exists a constant $c(\delta)$ such that*

$$\mathbf{P}\left[|n(r) - r^2| > \delta r^2\right] \leq e^{-c(\delta)r^4}.$$

When $\alpha > 2$, clearly $n((1-\delta)r^{\sqrt{\alpha}}) \geq n(r)$, whence from the above result it follows that

$$\mathbf{P}\left[n(r) \geq r^2 + r^\alpha\right] \leq \mathbf{P}\left[n((1-\delta)r^{\sqrt{\alpha}}) \geq r^\alpha\right]$$
$$\leq e^{-c(\delta)r^{2\alpha}}.$$

This gives

$$\limsup_{r\to\infty} \frac{\log\log\left(\frac{1}{\mathbf{P}[|n(r)-r^2|>r^\alpha]}\right)}{\log r} \leq 2\alpha. \tag{13}$$

The obviously loose inequality $n((1-\delta)r^{\sqrt{\alpha}}) \geq n(r)$ that we used, suggests that (13) can be improved when $\alpha > 2$ to Theorem 3.

Proof of Theorem 3: Lower bound. Let $m = r^2 + \gamma r^\alpha$. Suppose the mth term dominates the sum of all the other terms on $\partial D(0; r)$, i.e., suppose

$$\left| \frac{a_m z^m}{\sqrt{m!}} \right| \geq \left| \sum_{n\neq m} \frac{a_n z^n}{\sqrt{n!}} \right| \quad \text{whenever } |z| = r. \tag{14}$$

Now we want to find a lower bound for the probability of the event in (14). Note that the left side of (14) is identically equal to $\frac{|a_m| r^m}{\sqrt{m!}}$.

Now suppose the following happen-

1. $|a_n| \leq n \ \forall n \geq m+1$.
2. $|a_m| \geq m$.

3. $|a_n|\frac{r^n}{\sqrt{n!}} < \frac{\gamma r^\alpha}{m}\frac{r^m}{\sqrt{m!}}$ for every $0 \le n \le m-1$.

Then the right hand side of (14) is bounded by

$$\begin{aligned}
\text{RHS of (14)} &\le \sum_{n=0}^{m-1} |a_n|\frac{r^n}{\sqrt{n!}} + \sum_{n=m+1}^{\infty} \frac{|a_n| r^n}{\sqrt{n!}} \\
&\le \sum_{n=0}^{m-1} \frac{\gamma r^\alpha}{m}\frac{r^m}{\sqrt{m!}} + \sum_{n=m+1}^{\infty} \frac{n r^n}{\sqrt{n!}} \\
&\le \frac{m r^m}{\sqrt{m!}}\left(\frac{\gamma r^\alpha}{m} + 1 + o(1)\right) \\
&\le |a_m|\frac{r^m}{\sqrt{m!}}.
\end{aligned}$$

Thus if the above three events occur, then the mth term dominates the sum of all the other terms on $\partial D(0; r)$. Also these events have probabilities as follows.

1. $\mathbf{P}[|a_n| \le n \ \forall n \ge m+1] \ge 1 - \sum_{n=m+1}^{\infty} e^{-n^2} \ge 1 - C' e^{-m^2} = 1 - o(1)$.
2. $\mathbf{P}[|a_m| \ge m] = e^{-m^2} = e^{-\gamma^2 r^{2\alpha}(1+o(1))}$.
3. The third event has probability as follows. Recall again that $\mathbf{P}[\xi < x] \ge \frac{x}{2}$ if $x < 1$ and ξ is Exponential with mean 1. We apply this below with $x = \frac{\gamma^2 r^{2\alpha}}{m^2}\frac{r^{2m-2n} n!}{m!}$. This is clearly less than 1 if $n \ge r^2$. Therefore if r is sufficiently large it is easy to see that for all $0 \le n \le m-1$, the same is valid.
   Thus

$$\begin{aligned}
\mathbf{P}\left[|a_n| \le \frac{\gamma r^\alpha}{m}\frac{r^{m-n}\sqrt{n!}}{\sqrt{m!}} \ \ \forall n \le m-1\right] &= \prod_{n=0}^{m-1} \mathbf{P}\left[|a_n| \le \frac{\gamma r^\alpha}{m}\frac{r^{m-n}\sqrt{n!}}{\sqrt{m!}}\right] \\
&\ge \prod_{n=0}^{m-1} \frac{\gamma^2 r^{2\alpha}}{m^2}\frac{r^{2m-2n} n!}{2m!} \\
&= r^{2\alpha(m+1)+m(m+1)} 2^{-m} m^{-2m} e^{-\sum_{k=1}^{m} k\log k} \\
&= e^{m^2\log(r) - \frac{1}{2}m^2\log(m) + O(m^2)} \\
&= e^{-(\frac{\alpha}{2}-1)\gamma^2 r^{2\alpha}\log(r) + O(r^{2\alpha})}
\end{aligned}$$

Since these three events are independent, we get

$$\mathbf{P}\left[n(r) \geq r^2 + r^\alpha\right] \geq e^{-\left(\frac{\alpha}{2}-1\right)\gamma^2 r^{2\alpha}\log r + O(r^{2\alpha})}. \tag{15}$$

Upper bound. We omit the proof of the upper bound, as it follows the same lines as that of Theorem 1 and we have already seen such arguments again in the proof of Theorem 2 (In those two cases as well as the present case, we are looking at very large deviations, and that is the reason why the same tricks work).

Moreover note that the lower bound along with (13) proves the statement in the conjecture. □

Case $1 < \alpha < 2$**:** We prove Theorem 4. Along with Theorem 3 this shows that the asymptotics of $\mathbf{P}\left[n(r) \geq r^2 + \gamma r^\alpha\right]$ does undergo a qualitative change at $\alpha = 2$.

Proof of Theorem 4: Write $m = r^2 + \gamma r^\alpha$. As usual, we bound $\mathbf{P}\left[n(r) \geq m\right]$ from below by the probability of the event that the mth term dominates the rest of the series.

Firstly, we need a couple of estimates. Consider $\frac{r^{2n}}{n!}$ as a function of n. This increases monotonically up to $n = r^2$ and then decreases monotonically. $m = r^2 + \gamma r^\alpha$ is on the latter part. Write $M = r^2 - \gamma r^\alpha$.

Firstly, observe that $(r^2 - k)(r^2 + k) < (r^2)^2$, for $1 \leq k \leq \gamma r^\alpha$, whence $r^{2m-2M} > \prod\limits_{j=M+1}^{m-1} j$. This implies that

$$\frac{r^M}{\sqrt{M!}} < \frac{r^m}{\sqrt{m!}} \tag{16}$$

Secondly, note that for any $n = M - p$,

$$\begin{aligned}
\frac{r^{2n}/n!}{r^{2M}/M!} &= \prod_{j=0}^{p-1} \frac{M-j}{r^2} \\
&= \prod_{j=0}^{p-1} (1 - \gamma r^{\alpha-2} - jr^{-2}) \\
&\leq e^{-\sum\limits_{j=0}^{p-1}(\gamma r^{\alpha-2} + jr^{-2})} \\
&= e^{-\gamma p r^{\alpha-2} - \frac{p(p-1)}{2} r^{-2}}.
\end{aligned}$$

Now we set $p = Cr^{2-\alpha}$ with C so large that $e^{-\gamma C} \leq \frac{1}{4}$.

Then also note that if $n < M - kp$, it follows that

$$\frac{r^{2n}/n!}{r^{2m}/m!} \le \frac{1}{4^k}, \tag{17}$$

where we used (16) to replace M by m.

Thirdly, if $n = m + p$ with $p \le r^2 - \gamma r^\alpha$, then,

$$\begin{aligned}\frac{r^{2n}/n!}{r^{2m}/m!} &= \prod_{j=1}^{p} \frac{r^2}{m+j} \\ &= \prod_{j=1}^{p} (1 + \gamma r^{\alpha-2} + jr^{-2})^{-1} \\ &\le e^{-\frac{1}{2}\sum_{j=1}^{p}(\gamma r^{\alpha-2} + jr^{-2})} \\ &= e^{-\frac{1}{2}(\gamma p r^{\alpha-2} + \frac{p(p+1)}{2} r^{-2})}.\end{aligned}$$

If $p = 2Cr^{2-\alpha}$, where C was as chosen before, then for $n > m + kp$, we get

$$\frac{r^{2n}/n!}{r^{2m}/m!} \le \frac{1}{4^k}. \tag{18}$$

From now on $p = 2Cr^{2-\alpha}$ is fixed so that (17) and (18) are satisfied.

Next we divide the coefficients other than m into groups:

- $A_k = \{n : n \in (M - kp, M - (k-1)p] \quad \text{for } 1 \le k \le \lceil \frac{M}{p} \rceil\}$.
- $D_k = \{n : n \in [m + (k-1)p, m + kp) \quad \text{for } 1 \le k \le \lceil \frac{M}{p} \rceil\}$.
- $B = \{n : n \in [M+1, m-1]\}$.
- $C = \{n : n \in [2r^2, \infty)\}$.

Remark 11. *As defined, there is an overlap between $D_{\lceil \frac{M}{p} \rceil}$ and C. This is inconsequential, but for definiteness, let us truncate the former interval at $2r^2$ (just as $A_{\lceil \frac{M}{p} \rceil}$ is understood to be truncated at 0).*

Now consider the following events.

1. $|a_n| \le \frac{2^k}{M} \quad \text{for } n \in A_k \quad \text{for } k \le \lceil \frac{M}{p} \rceil \}$.
2. $|a_n| \le \frac{2^k}{M} \quad \text{for } n \in D_k \quad \text{for } k \le \lceil \frac{M}{p} \rceil \}$.
3. $\sum_{n \in B} |a_n| \frac{r^n}{\sqrt{n!}} \le 4 \frac{r^m}{\sqrt{m!}}$.
4. $|a_n| < n - 2r^2 \quad \text{for } n \in C$.
5. $|a_m| \ge 15$.

Suppose all these events occur. Then

1. The event $|a_n| \le \frac{2^k}{M}$ for $n \in A_k$, $k \le \lceil \frac{M}{p} \rceil$ gives

$$\sup\left\{ \left| \sum_{n=0}^{M} \frac{a_n z^n}{\sqrt{n!}} \right| : |z| = r \right\} \le \sum_{k=1}^{\lceil M/p \rceil} \sum_{n=M-kp+1}^{M-(k-1)p} |a_n| \frac{r^n}{\sqrt{n!}} \tag{19}$$

$$\le \sum_{k=1}^{\lceil M/p \rceil} \frac{1}{2^k} \frac{r^m}{\sqrt{m!}} \frac{2^k p}{M} \quad \text{by (17)} \tag{20}$$

$$\le \frac{r^m}{\sqrt{m!}} \sum_{k=1}^{\lceil M/p \rceil} \frac{p}{M} \tag{21}$$

$$\le \frac{r^m}{\sqrt{m!}} \left(1 + \frac{p}{M}\right). \tag{22}$$

2. The event $|a_n| \le \frac{2^k}{M}$ for $n \in D_k$, $k \le \lceil \frac{M}{p} \rceil$ gives

$$\sup\left\{ \left| \sum_{n=m+1}^{2r^2} \frac{a_n z^n}{\sqrt{n!}} \right| : |z| = r \right\} = \sum_{k=1}^{\lceil M/p \rceil} \sum_{n=m+(k-1)p+1}^{M+kp} |a_n| \frac{r^n}{\sqrt{n!}} \tag{23}$$

$$\le \sum_{k=1}^{\lceil M/p \rceil} \frac{1}{2^k} \frac{r^m}{\sqrt{m!}} \frac{2^k p}{M} \quad \text{by (18)} \tag{24}$$

$$\le \frac{r^m}{\sqrt{m!}} \left(1 + \frac{p}{M}\right). \tag{25}$$

3. The third event gives

$$\sum_{n \in B} |a_n| \frac{r^n}{\sqrt{n!}} \le 4 \frac{r^m}{\sqrt{m!}}, \tag{26}$$

   by assumption.
4. The event $|a_n| < n - 2r^2$ for $n \in C$: Since $n > 2r^2$,

$$\begin{aligned} \frac{r^n}{\sqrt{n!}} &= \frac{r^m}{\sqrt{m!}} \prod_{k=m+1}^{n} \frac{r}{\sqrt{k}} \\ &\le \frac{r^m}{\sqrt{m!}} \prod_{k=2r^2+1}^{n} \frac{r}{\sqrt{k}} \\ &\le \frac{r^m}{\sqrt{m!}} \left(\frac{1}{\sqrt{2}}\right)^{n-2r^2}. \end{aligned}$$

Therefore we get (using $|a_n| < n - 2r^2 \ \forall n > 2r^2$)

$$\sum_{n\in C} |a_n| \frac{r^n}{\sqrt{n!}} \le \frac{r^m}{\sqrt{m!}} \sum_{n>2r^2} (n-2r^2)\left(\frac{1}{\sqrt{2}}\right)^{n-2r^2} \tag{27}$$

$$= \frac{r^m}{\sqrt{m!}} \frac{\sqrt{2}}{(\sqrt{2}-1)^2}. \tag{28}$$

Putting together the contributions from these four groups of terms, and using $|a_m| > 15$, we get (for large values of r)

$$\sum_{n\neq m} |a_n| \frac{r^n}{\sqrt{n!}} \le |a_m| \frac{r^m}{\sqrt{m!}}.$$

Now we compute the probabilities of the events enumerated above.

1. The event $|a_n| \le \frac{2^k}{M}$ for $n \in A_k$ for $k \le \lceil \frac{M}{p} \rceil$. Now for a fixed $k \le 3\log_2(r)$, we deduce

$$\mathbf{P}\left[|a_n| \le \frac{2^k}{M} \quad \text{for } n \in A_k\right] \ge \mathbf{P}\left[|a_n| \le \frac{1}{M} \quad \text{for } n \in A_k\right]$$
$$\ge \left(\frac{1}{2M^2}\right)^p.$$

Therefore

$$\mathbf{P}\left[|a_n| \le \frac{2^k}{M} \quad \text{for } n \in A_k \quad \text{for every } k \le 3\log_2(r)\right] \ge \left(\frac{1}{2M^2}\right)^{3p\log_2(r)}$$
$$\ge e^{-cr^{2-\alpha}(\log(r))^2}.$$

for some c.

Next we deal with $k > 3\log_2(r)$.

$$\mathbf{P}\left[|a_n| \le \frac{2^k}{M} \quad \text{for } n \in A_k \quad \text{for every } k > 3\log_2(r)\right]$$
$$\ge 1 - \sum_{k>3\log_2(r)} p\mathbf{P}\left[|a| > \frac{2^k}{M}\right]$$
$$= 1 - \sum_{k>3\log_2(r)} pe^{-2^{2k}M^{-2}}.$$

Now the summation in the the last line has rapidly decaying terms and starts with $pe^{-2^{6\log_2(r)}M^{-2}}$ which is smaller than pe^{-r^2}. Thus

$$\mathbf{P}\left[|a_n| \leq \frac{2^k}{M} \quad \text{for } n \in A_k \quad \text{for every } k > 3\log_2(r)\right] = 1 - o(1).$$

Thus the event in question has probability at least $e^{-cr^{2-\alpha}(\log(r))^2(1+o(1))}$.

2. The event $|a_n| \leq \frac{2^k}{M}$ for $n \in D_k$ for $k \leq \lceil \frac{M}{p} \rceil$. Following exactly the same steps as above we can prove that

$$\mathbf{P}\left[|a_n| \leq \frac{2^k}{M} \quad \text{for } n \in D_k\right] \geq e^{-cr^{2-\alpha}(\log(r))^2(1+o(1))}.$$

3. The event $\sum\limits_{n\in B} |a_n|\frac{r^n}{\sqrt{n!}} \leq 4\frac{r^m}{\sqrt{m!}}$. By Cauchy-Schwarz, $(\sum_{n\in B} |a_n|\frac{r^n}{\sqrt{n!}})^2 \leq (\sum_{n\in B} |a_n|^2)(\sum_{n\in B} \frac{r^{2n}}{n!})$. $Y = \sum_{n\in B} |a_n|^2$ has $\Gamma(|B|, 1)$ distribution. Also $\sum_{n\in B} \frac{r^{2n}}{n!} \leq e^{r^2}$, since the left hand is part of the Taylor series of e^{r^2}. Therefore the event in question has probability,

$$\begin{aligned}\mathbf{P}[\text{event in question}] &\geq \mathbf{P}\left[Y < 16\frac{r^{2m}}{m!}e^{-r^2}\right]\\ &\geq \varphi\left(8\frac{r^{2m}}{m!}e^{-r^2}\right)8\frac{r^{2m}}{m!}e^{-r^2},\end{aligned}$$

where φ is the density of the $\Gamma(|B|, 1)$ distribution. This last follows because φ is increasing on $[0, |B|]$ and thus $\mathbf{P}[Y < x] \geq \varphi\left(\frac{x}{2}\right)\frac{x}{2}$. for $x < |B|$. Continuing,

$$\mathbf{P}[\text{event in question}] \geq \frac{1}{(2\gamma r^\alpha)!}e^{-8\frac{r^{2m}}{m!}e^{-r^2}}\left(8\frac{r^{2m}}{m!}e^{-r^2}\right)^{2\gamma r^\alpha} \tag{29}$$

$$\geq Ce^{2\gamma r^\alpha m\log(r^2) - 2\gamma r^{2+\alpha} - 2\gamma r^\alpha m\log(m) + 2\gamma r^\alpha m + O(r^\alpha \log(r))}. \tag{30}$$

where we used Stirling's approximation.

The exponent needs simplification. Take the first and third terms in the exponent. We have $-2\gamma m r^\alpha \log\left(\frac{m}{r^2}\right)$. Recall that $m = r^2 + \gamma r^\alpha$ and that $\alpha < 2$. Therefore by Taylor's expansion of $\log(1 + \gamma r^{\alpha-2})$ we get

$$-2\gamma m r^\alpha \log\left(\frac{m}{r^2}\right) = \left\{\begin{array}{llll} -2\gamma^2 r^{2\alpha} & +\gamma^3 r^{3\alpha-2} & -\frac{2}{3}\gamma^4 r^{4\alpha-4} & + \cdots \\ -2\gamma^3 r^{3\alpha-2} & +\gamma^4 r^{4\alpha-4} & -\frac{2}{3}\gamma^5 r^{5\alpha-6} & + \cdots \end{array}\right. \tag{31}$$

Now consider (30). Expand the fourth term in the exponential as $2\gamma r^{2+\alpha} + 2\gamma^2 r^{2\alpha}$. We get the following terms

- $r^{2+\alpha}(-2\gamma + 2\gamma) = 0$, from the second and fourth terms (first piece of the fourth term) in the exponential in (30).

- $r^{2\alpha}(-2\gamma^2 + 2\gamma^2) = 0$, from the sum of the first term in the expansion (31) and the second piece of the fourth term in the exponential in (30).
- $r^{3\alpha-2}(\gamma^3 - 2\gamma^3) = -\gamma^3 r^{3\alpha-2}$, from the expansion (31).
- Other terms such as $r^\alpha \log(m)$, $r^\alpha \log(r)$, r^α, $r^{4\alpha-4}$, $r^{5\alpha-6}$ etc. All these are of lower order than $r^{3\alpha-2}$ when $1 < \alpha < 2$.

Hence,

$$\mathbf{P}[\text{event in quesion}] \geq e^{-\gamma^3 r^{3\alpha-2}(1+o(1))}.$$

4. The event $|a_n| < n - 2r^2$ for $n \in C$. This is just an event for a sequence of i.i.d. Complex Gaussians. It has a fixed probability p_0 (say).
5. The event $|a_m| \geq 15$ also has a constant probability (not depending on r, that is).

This completes the estimation of probabilities. Among these five events, the third one, namely $\sum\limits_{n\in B} |a_n|\frac{r^n}{n!} \leq 4\frac{r^m}{\sqrt{m!}}$ has the least probability (Recall that $1 < \alpha < 2$).

Also these events are all independent, being dependent on disjoint sets of coefficients. Thus $\mathbf{P}[n(r) \geq r^2 + \gamma r^\alpha] \geq e^{-\gamma^3 r^{3\alpha-2}(1+o(1))}$. □

ACKNOWLEDGMENTS

I am grateful to Yuval Peres and Mikhail Sodin for suggesting the problem of overcrowding and the problem of moderate deviations respectively. I also thank them for innumerable discussions which illuminated many aspects of two problems considered here. Earlier I had proved the lower bound in Theorem 4 only for $\alpha > \frac{4}{3}$. I am indebted to Fedor Nazarov for showing me how to prove (communicated to me through Misha Sodin) the bound for all $\alpha > 1$.

REFERENCES

1. M. Sodin, Zeroes of Gaussian analytic functions *European Congress of Mathematics, Stockholm, June 27–July 2* (2004).
2. M. Sodin and B. Tsirelson, Random complex zeroes. I. Asymptotic normality. *Israel J. Math.* **144**:125–149 (2004).
3. M. Sodin, Zeroes of Gaussian analytic functions. *Math. Res. Lett* **7(4)**:371–381 (2000).
4. B. Jancovici, J. L. Lebowitz and G. Manificat, Large charge fluctuations in classical Coulomb systems. *J. Statist. Phys.* **72**:773–787 (1993).
5. M. Sodin and B. Tsirelson, Random Complex Zeroes. III. Decay of the hole probability. *Israel J. Math.* **147**:371–379 (2005).
6. A. Soshnikov, Determinantal random point fields. *Russian Math. Surveys* **55**:923–975 (2000).
7. B. Hough, M. Krishnapur, Y. Peres and B. Virág, Determinantal Processes and Independence. *Probab. Surv.* **3**:206–229 (2006).
8. J. Ginibre, Statistical ensembles of complex, quaternion, and real matrices. *J. Mathematical Phys.* **6**:440–449 (1965).

9. M. R. Dennis and J. H. Hannay, Saddle points in the chaotic analytic function and Ginibre characteristic polynomial. *J. Phys. A* **12**:3379–3383 (2003).
10. E. Kostlan, On the spectra of Gaussian matrices. *Linear Algebra Appl.* **162/164**:385–388 (1992).
11. Y. Peres and B. Virág, Zeros of i.i.d. Gaussian power series: a conformally invariant determinantal process. *Acta Mathematica* **194**:1–35 (2005).